№474

$$u=\frac{f(x-at)+f(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}F(z)dz$$

Получаем:

$$u(x,t)=\frac{\cos(x-at)+\cos(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\omega z dz=\frac{1}{2}(\cos(x-at)+\cos(x+at))+\frac{\omega}{4a}*z^2\Big|_{x-at}^{x+at}=$$

$$=\frac{1}{2}(\cos(x-at)+\cos(x+at))+\frac{\omega}{4a}((x+at)^2-(x-at)^2)=\frac{1}{2}(\cos(x-at)+\cos(x+at))+\omega t.$$

№484

$$\omega=e^{1-2z},\quad z_0=\frac{\pi i}{3}$$

Воспользуемся формулой:

$$e^{-iz}=\cos z+i\sin z$$

Получаем:

$$\omega=e^{1-2z}=e*e^{-2z}=e*e^{-2x-2iy}=e^{1-2x}*e^{-2iy}=e^{1-2x}(\cos 2y-i\sin 2y)$$

$$u=e^{1-2x}\cos 2y$$

$$V=-e^{1-2x}\sin 2y$$

Проверим условие Коши - Римана:

$$\frac{\partial u}{\partial x}=-2e^{1-2x}\cos 2y=\frac{\partial V}{\partial y}=-2e^{1-2x}\cos 2y$$

$$-\frac{\partial V}{\partial x}=-e^{1-2x}*\sin 2y=\frac{\partial u}{\partial y}=-2e^{1-2x}\sin 2y$$

Следовательно, функция является аналитической. Найдём её производную в точке $z_0=\frac{\pi i}{3}$.

$$f'(z)=-2e^{1-2z}$$

$$f'(z_0)=f'\left(\frac{\pi i}{3}\right)=-2e^{1-\frac{2\pi i}{3}}=-2ee^{-\frac{2\pi i}{3}}=-2e\left(\cos\left(-\frac{2\pi}{3}\right)+\sin\left(-\frac{2\pi}{3}\right)\right)=-2e\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=e+ie\sqrt{3}.$$

№494

$$f(z)=\frac{1}{(z^2+1)^2},\quad z_0=i$$

$$f(z)=\frac{1}{(z^2+1)^2}=\frac{1}{(z-i)^2(z+i)^2}=\frac{A}{z-i}+\frac{B}{(z-i)^2}+\frac{C}{z+i}+\frac{D}{(z+i)^2}=$$

$$=\frac{(Az+Ai)(z^2+i)^2+B(z+i)^2+(Cz+Ci)(z-i)^2+D(z-i)^2}{(z-i)^2(z+i)^2}$$

$$\begin{array}{l|c}
z^3 & A+C=0 \\
z^2 & -Ai+2Ai+C+D-2Ci+Ci=0 \\
z^1 & -A+2A+2Bi-2Di+2C-C=0 \\
z^0 & Ai-B-Ci-D=1
\end{array}$$

Находим:

$$A=-\frac{i}{4};\ \ B=-\frac{1}{4};\ C=\frac{i}{4};\quad D=-\frac{1}{4}.$$

$$f(z)=-\frac{i}{4(z-i)}-\frac{1}{4(z-i)^2}+\frac{i}{4}\frac{1}{(z+i)}-\frac{1}{4}\frac{1}{(z+i)^2}$$

$$\frac{1}{z+i}=\frac{1}{2i\left(1-\left(\dfrac{z-i}{-2i}\right)\right)}=\frac{1}{2i}\sum_{n=0}^{\infty}\frac{(z-i)^n}{(-2i)^n}$$

$$\frac{1}{(z+i)^2}=-\frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{(z-i)^n}{(-2i)^n}\right)^2$$

$$f(z)=-\frac{i}{4(z-i)}-\frac{1}{4(z-i)^2}+\frac{1}{8}\sum_{n=0}^{\infty}\frac{(z-i)^n}{(-2i)^n}+\frac{1}{16}\left(\sum_{n=0}^{\infty}\frac{(z-i)^n}{(-2i)^n}\right)^2.$$

№504

$x''-9x=e^{-2t};\quad x(0)=0;\quad x'(0)=0$

Переходим к изображениям:

$x \leftarrow x(p)$

$$p^2\bar{x}=px(0)-x'(0)-9\bar{x}=\frac{1}{p+2}$$

$$p^2\bar{x}-9\bar{x}=\frac{1}{p+2}$$

$$\bar{x}(p^2-9)=\frac{1}{p+2}$$

$$\bar{x}=\frac{1}{(p+2)(p^2-9)}=\frac{A}{p+2}+\frac{B}{p-3}+\frac{C}{p+3}=\frac{A(p^2-9)+B(p^2+5p+6)+C(p^2-p-6)}{(p+2)(p^2-9)}$$

$$\begin{array}{l|l} p^2 & A+B+C=0 \qquad\quad C=5B \\ p^1 & 5B-C=0 \qquad \Rightarrow A+6B=0 \\ p^0 & -9A+6B-6C=1 \qquad A=-6B \end{array}$$

$$54B+6B-30B=1$$

$$30B=1\Rightarrow B=\frac{1}{30}$$

$$C=\frac{1}{6};\quad A=-\frac{1}{5}$$

$$\bar{x}=-\frac{1}{5(p+2)}+\frac{1}{30(p-3)}+\frac{1}{6(p+3)},\quad \text{откуда:}$$

$$x=-\frac{1}{5}e^{-2t}+\frac{1}{30}e^{3t}+\frac{1}{6}e^{-3t}.$$

№514

$$\begin{cases} x'+y-z=0 \\ y'-z=0 \\ x+z-z'=0 \end{cases} \qquad x(0)=2 \quad y(0)=\frac{1}{2} \quad z(0)=\frac{5}{2}$$

$$x \leftarrow x(p) \qquad x'(p) \leftarrow px(p)-x(0)=px(p)-2$$

$$y \leftarrow y(p) \qquad y'(p) \leftarrow py(p)-y(0)=py(p)-\frac{1}{2}$$

$$z \leftarrow z(p) \qquad z'(p) \leftarrow pz(p)-z(0)=pz(p)-\frac{5}{2}$$

$$\begin{cases} px(p)-2+y(p)-z(p)=0 \\ py(p)-\frac{1}{2}-z(p)=0 \\ x(p)+z(p)-pz+\frac{5}{2}=0 \end{cases} \Bigg| (-1)$$

$$x(p)=(p-1)*z(p)-\frac{5}{2}$$

$$y(p)=\frac{1}{2p}+\frac{z(p)}{p}$$

$$p(p-1)*z(p)-\frac{5}{2}*p-2+\frac{1}{2p}+\frac{z(p)}{p}-z(p)=0$$

$$z(p)\left(p^2-p+\frac{1}{p}-1\right)=\frac{5}{2}p+2-\frac{1}{2p}$$

$$z(p)=\frac{\frac{5p^2+4p-1}{2p}}{\frac{p^3-p^2+1-p}{p}}=\frac{5p^2+4p-1}{2(p^3-p^2+1-p)}=\frac{5p^2+4p-1}{2(p^2(p-1)-1(p-1))}=\frac{5p^2+4p-1}{2(p-1)(p^2-1)}=$$

$$=\frac{5p^2+4p-1}{2(p-1)^2(p+1)};$$

$$x(p)=(p-1)\frac{p^2+4p-1}{2(p-1)^2(p+1)}-\frac{5}{2}=\frac{p^2+4p-1}{2(p-1)(p+1)}-\frac{5}{2}=\frac{p^2+4p-1-5p^2+5}{2(p-1)(p+1)}=\frac{-4p^2+4p+4}{2(p-1)(p+1)}=$$

$$=\frac{-2p^2+2p+4}{(p-1)(p+1)}=\frac{-2(p^2-1)+4}{(p-1)(p+1)}=-2+\frac{4}{(p-1)(p+1)};$$

$$y(p)=\frac{1}{2p}+\frac{\frac{5p^2+4p-1}{2(p-1)^2(p+1)}}{p}=\frac{(p-1)^2(p+1)+5p^2+4p-1}{2p(p-1)^2(p+1)}=\frac{(p^2-1)(p-1)+5p^2+4p-1}{2p(p-1)^2(p+1)}=$$

$$=\frac{p^3-p^2-p+1+5p^2+4p-1}{2p(p-1)^2(p+1)}=\frac{p^3+4p^2+3p}{2p(p-1)^2(p+1)}=\frac{p^2+4p+3}{2(p-1)^2(p+1)};$$

$$\frac{4}{(p-1)(p+1)}=\frac{A}{p-1}+\frac{B}{p+1}=\frac{Ap+a+Bp-B}{(p-1)(p+1)}=\frac{p(A+B)+(A-B)}{(p-1)(p+1)}.$$

$$\begin{array}{l|l} p^1 & A+B=0 \\ p^0 & A-B=4 \end{array} \qquad \begin{array}{c} 2A=4 \\ A=2 \end{array} \qquad B=-A=-2$$

$$\frac{4}{(p-1)(p+1)}=\frac{2}{p-1}-\frac{2}{p+1}$$

$$\frac{p^2+4p+3}{(p-1)^2(p+1)}=\frac{A}{p-1}+\frac{B}{(p-1)^2}+\frac{C}{p+1}=\frac{A(p-1)(p+1)+B(p+1)+C(p-1)^2}{(p-1)^2(p+1)}=$$

$$=\frac{Ap^2-A+Bp+B+Cp^2-2Cp+C}{(p-1)^2(p+1)}=\frac{p^2(A+C)+p(B-2C)+(-A+B+C)}{(p-1)^2(p+1)};$$

$$\begin{array}{l|c} p^2 & A+C=1 \\ p^1 & B-2C=4 \\ p^0 & -A+B+C=3 \end{array} \qquad B+2C=4 \qquad \begin{array}{c} 2\mathrm{B}=4 \\ B=2 \end{array}$$

$$2-2C=4$$

$$-2C=2 \qquad A-1=1$$

$$C=-1 \qquad A=2$$

$$\frac{p^2+4p+3}{(p-1)^2(p+1)}=\frac{2}{p-1}+\frac{2}{(p-1)^2}+\frac{-1}{p+1}$$

$$y(p)=\frac{1}{2}\left(\frac{2}{p-1}+\frac{2}{(p-1)^2}+\frac{-1}{p+1}\right)=\frac{1}{p-1}+\frac{1}{(p-1)^2}-\frac{1}{2(p+1)};$$

$$\frac{5p^2+4p-1}{(p-1)^2(p+1)}=\frac{A}{p-1}+\frac{B}{(p-1)^2}+\frac{C}{p+1}$$

$$\begin{array}{l|c|} p^2 & A+C=5 \\ p^1 & B-2C=4 \\ p^0 & -A+B+C=3 \end{array} \begin{array}{c} 2B+2C=9 \\ B-2C=4 \end{array}$$

$$2B=12$$

$$B=6$$

$$6-2C=4$$

$$-2C=-2$$

$$C=1$$

$$A+1=5$$

$$A=4$$

$$z(p)=\frac{1}{2}\left(\frac{4}{p-1}+\frac{6}{(p-1)^2}+\frac{1}{p+1}\right)=\frac{2}{p-1}+\frac{3}{(p-1)^2}+\frac{1}{2(p+1)}$$

$$x=-2+\frac{2}{p-1}-\frac{2}{p+1}; \qquad x(t)=-2+2e^t-2e^{-t}$$

$$y(p)=\frac{1}{p-1}+\frac{1}{(p-1)^2}-\frac{1}{2(p+1)}$$

$$y(t)=e^t+e^t*t-\frac{1}{2}e^{-t}$$

$$z(t)=2e^t+3te^t+\frac{1}{2}e^{-t}.$$